RAF 34.

CONFIDENTIAL

~~SECRET~~

AIR MINISTRY

GREAT BADDOW

RESEARCH REPORT

TR.477

THE EFFECT OF THE GEOMAGNETIC FIELD ON ELECTRO - MAGNETIC WAVES VERTICALLY INCIDENT ON THE IONOSPHERE.

APRIL 1944

Air Ministry,
Great Baddow Research Station,
Nr. Chelmsford,
Essex.

CONFIDENTIAL

Ref. TR/477.

J.W. COX. December, 1943

THE EFFECT OF THE GEOMAGNETIC FIELD ON ELECTROMAGNETIC WAVES VERTICALLY INCIDENT ON THE IONOSPHERE

SUMMARY.

A numerical investigation is made of the effect of the geomagnetic field on the apparent height, attenuation and lateral deviation of waves vertically incident on the ionosphere. The "ordinary" wave is particularly considered, and it is shown that contrary to what is often assumed, the overall effects of the field on the path are considerable. Moreover this overall effect is not produced by a small change in the mode of propagation, but is the final result of large changes of opposite sign in different parts of the path. The algebra is interpreted in physical terms and some estimation is made of the probable effect on long-distance transmission.

INTRODUCTION.

The differential equations which govern the motion of wave-packets in a stratified ionized medium in the presence of a steady magnetic field are well known, but there has been little detailed numerical evaluation of the effect on observable quantities which may be expected to be produced by the field. The observables are the group time, the relation in space between the entering and emergent paths, the attenuation, and the polarization. Many articles have shown the general course of the differentials of these quantities (e.g. refractive index) but in almost all cases the lack of integration has led the authors to overlook certain features of the differentials which are of considerable practical importance. In this article the integration of certain cases is performed numerically and the observable quantities calculated. The results obtained agree with those of Millington[1] so far as they overlap, but are carried out in greater detail. It is shown that in many cases where it has always been assumed that the effect of the field is negligible, it is in fact pronounced and its magnitude is determined.

PROGRAMME.

The general problem of the magneto-ionic effect contains so many variables, and the algebra is so complicated, that it has not been found possible to obtain useful general expressions for the observable effect. Instead, a specific problem of considerable practical importance has been worked out in detail.

The usual assumptions are made, i.e.

(a) The ionosphere is stratified vertically only.

(b) The density gradients are small, so that a ray-treatment is valid.

(c) The collision frequency is appreciable but small compared with the wave-frequency.

Of the quanties already listed, the path time and attenuation have been calculated. The polarization at any level could be found but the observable polarization is determined at the bottom of the layer only, and there is already in existence a comprehensive graphical method of computing it[2].

The relation between incident and emergent packets is simple as they both travel vertically through the same point but the lateral deviation at any level, the overall effect of which is expressed by the relation in the general case, has been computed instead as it is found to be an important quantity in the physical interpretation.

On this basis the behaviour of the ordinary ray at vertical incidence in a region where the magnetic dip is -67^{o} and the gyromagnetic frequency 1.2 Mc/s is evaluated. This value of dip and frequency corresponds to conditions in England and most of Western Europe. The ordinary ray has been investigated because it has so often been assumed that the effect of the field will be negligible, although this is known to be strictly true only at the magnetic equator. In fact it is found that the effect of the field is considerable and easily within the observable range. Some of the consequences of this are discussed. It is realized that this ray-method is only a first approximation to the truth. It is however, accurate enough for many practical problems. Analytic continuation is made to certain cases where the method is physically invalid, in order to provide checks on graphical methods by means of analytical solutions.

PART I. DIFFERENTIALS.

1. Analysis.

The method adopted in this work has been to calculate the differential quantities, group velocity, direction of motion of wave-packet, and rate of attenuation, as functions of frequency and electron density. They are then integrated through layers chosen to correspond roughly with some typical ionospheric conditions.

Expressions for all these quantities at vertical incidence are well known, but numerical work has been hampered by the fact that their complications makes them difficult to compute. The methods of computation adopted here are based on analysis in course of publication developed by Eckersley and Millington to deal with the general case of oblique incidence. They are therefore given without proof in the form which they take at vertical incidence, as it is not obvious in this special case just why such methods should suggest themselves. It is however easy to see that they do agree with the known expressions and are also numerically convenient. The work is expressed in terms of the motion of a wave packet limited in space as well as in time, whose direction of motion outside the ionosphere is vertical, whereas most detailed vertical incidence treatments are in terms of an infinite plane wave vertically incident. The observable results are the same, but it is found much simpler to interpret the algebra in physical terms by considering a limited packet. The notation used, together with certain auxiliary functions, in finding the differentials, is:-

f is the wave-frequency.
λ the wave-length in free space.
c the velocity in free space = 3×10^8 m/sec.
N the electronic density in electrons per cubic metre.
e the electronic charge in coulombs.
m the electronic mass in kilograms.
ε_o/μ_o the constants of free space.

$f_o = N e^2/4\pi^2 m\varepsilon_o$

H the geomagnetic field in weber/meter^2.

$f_H = \dfrac{e\mu_o H}{2\pi m}$

D the magnetic dip measured as negative in the northern hemisphere.

$\tau = f_H/f$.

$\zeta = f_o^2/f^2$.

f_c the collision frequency of the electrons.
β a constant of the order 3/2.

$\alpha = f_c/\beta\pi f$ and is assumed small compared with unity.

z the height above the bottom of the layer.
u horizontal distance in the meridian plane from the point of entry into the layer.
Z the refractive index.
cU the vertical component of the group velocity.
Y $Z \sin D$.
ℓ = the "Lorentz term" usually taken as 0 or 1/3.
$\eta = \zeta/(1 + \ell\zeta)$
$\sigma = \tau/(1 + \ell\zeta)$

$p = (1 - Z^2)/\eta$

$q = (1 - \eta)/\sigma^2$

$F_x = 1 \Big/ \left[\left(\dfrac{p^2 - 1}{2}\right) q - 1\right]$

$F_w = (p - 1)^2/p$

$\delta = 1 - \eta$

$A = 1 - \eta\sin^2 D$.

P' = the equivalent path length.

Further symbols and functions occur in the integration and will be defined as they arise.

The quantities F_x, F_w are best computed by eliminating the p in the definitions and expressing them as functions of Y and q. Graphs of these functions over the requisite range have been computed.

The fundamental equation is that connecting Z, σ, D, and δ (or η).

At vertical incidence this may be written

$$aZ^4 - bZ^2 + c = 0.$$

where $a = A\sigma^2 - \delta$

$$b = (A + \delta)^2 - 2\sigma^2$$

$$c = \delta\sigma^2 - \delta^3$$

a, b, and c may each be positive or negative, so that to avoid computing trouble use is made of the forms

Form 1. Form 2.

$$Z^2 = \frac{b + \sqrt{b^2 - 4\,ac}}{2a} \quad \text{or} \quad \frac{2c}{b - \sqrt{b^2 - 4\,ac}} \quad \text{(ordinary)}$$

$$= \frac{b + \sqrt{b^2 - 4\,ac}}{2a} \quad \text{or} \quad \frac{2c}{b + \sqrt{b^2 - 4\,ac}} \quad \text{(extraordinary)}$$

Form 2 leads directly to the usual form of the Appleton-Hartree equation.

The rule for computing is:-

Compute a, b, and c, and then work according to the sign of b.

	Ordinary	Extraordinary.
b positive	Form 2	Form 1.
b negative	Form 1.	Form 2.

Having found the Z - η relation, the differentials are computed from

$$\frac{du}{dz} = \frac{\eta F_x \sin D \cos D}{1 + \eta F_x \sin^2 D}$$

$$\frac{dP'}{dz} = \frac{1}{U} = \frac{1 + \eta F_x \left[Y^2 + \frac{F_w}{\sigma^2} \left\{ 1 - \ell(2 - \eta) \right\} \right] - \ell p}{Z\,(1 + \eta F_x \sin^2 D).}$$

The total attenuation is exp. (-K) where

$$K = \frac{2\pi}{\lambda} \int_0^z \frac{\alpha}{2} \eta k.\ dz$$

and
$$k = \frac{p + \cdot \frac{F_x F_w}{\sigma^2}\ (2 - \eta)}{Z\left[1 + \eta F_x \sin^2 D\right]} \quad (1 - \ell\eta)$$

It is evident that labour is saved by treating all three together, as similar quantities occur, especially in the denominators, i.e. if any one is to be computed, it is worth while to do all three. These differential quantities are now considered in detail. The computations have been arranged in such a way that the results can be graphically exhibited to allow a final accuracy of the order of 1% when the integration is made. This necessitates several full-page graphs, and to save space only copies showing the general shape of the graphs are reproduced.

During the analysis, reference will be made to various limiting values. These are collected here in the case $\ell = 0$.

At $\zeta = 1$

$$p = \frac{1 - Z^2}{\zeta} = 1.$$

$$q = \frac{1 - \zeta}{\tau^2} = 0.$$

$F_x = -1.$

$F_W = 0.$

At $\zeta = 0$

$$p \text{ is finite and } = \frac{2 - \tau^2\cos^2 D - \tau\sqrt{4\sin^2 D + \tau^2\cos^4 D}}{2(1 - \tau^2)}$$

$$q = \frac{1}{\tau^2}.$$

F_x and F_W have complicated values but can be computed.

At $\tau = 0$ (very high frequency or zero field).

$$p = \frac{1 - Z^2}{\zeta} = 1.$$

$$q = \frac{1 - \zeta}{\tau^2} \text{ and becomes infinite.}$$

$F_x = 0.$

$F_W = 0.$

Use is also made of $\tau = \infty$ (very low frequency). The results here will be physically incorrect, because the ray-theory cannot be used, but are useful as showing end-points towards which the curves must tend.

At $\tau = \infty$

$p = \cos^2 D.$

$q = 0.$

$F_x = -1,$

$F_W = \sin^4 D \sec^2 D.$

2. The $Z - \zeta$ Curves.

In the following detailed analysis it is assumed that $\ell = 0$.

The general shape to be expected of the $Z - \zeta$ curves is well known. $Z = 1$ for $\zeta = 0$ and $Z = 0$ for $\zeta = 1$. The curve is of a rather distorted shape changing very rapidly near $\zeta = 1$, and it is for this reason that the fundamental equation is best expressed in terms of δ. When $\tau = 0$ $Z = \sqrt{\delta}$. If $\tau \neq 0$, $Z > \sqrt{\delta}$ and when is small, the fundamental

equation becomes

$$Z^4 \cos^2 D - Z^2 \cos^2 D + \delta = 0$$

the appropriate root of which is

$$Z = \sqrt{\delta} \sec D.$$

Thus the quantity $Z/\sqrt{\delta}$ is always finite, and is conveniently plotted. As the distortion of the curve also occurs when δ is small, it is best to use a logarithmic scale of δ. The resultant curves for $Z/\sqrt{\delta}$ are shown in Fig.1. These are the fundamental curves on which the rest are based, and show up at once the distortion, already mentioned, at small values of δ. Imaginary values of Z are neglected. This disregards any coupling between different rays.

At zero frequency $\tau = \infty$

$$Z^2 = \frac{1 - \zeta}{1 - \zeta \sin^2 D}$$

$$\text{i.e.} \quad \frac{Z}{\sqrt{\delta}} = \frac{1}{\sqrt{A}}$$

3. The lateral deviation.

The tilt of the path is given by $\frac{du}{dz}$.

At $\zeta = 0$ $\frac{du}{dz} = 0$.

At $\zeta = 1$ $\frac{du}{dz} \to - \tan D$.

Thus the group-path tilts over till it is perpendicular to the magnetic field. The negative sign implies that the ordinary wave moves toward the north in the northern hemisphere.

At $\tau = 0$, $\frac{du}{dz} = 0$, as is to be expected.

At $\tau = \infty$ $\frac{du}{dz} = \frac{- \zeta \sin D \cos D}{1 - \zeta \sin^2 D} = \frac{- \zeta \sin 2D}{2A}$

Thus $\frac{du}{dz}$ is everywhere finite and there is no especial difficulty in representing it graphically as a function of δ. (Fig.2.). In Fig. 3 the same curves are represented as functions of f for definite values of

δ. This representation is not convenient when it is required to integrate on a frequency not represented in Fig. 2, but the physical picture is clearer from Fig. 2. All the differentials are shown here in the form of Fig. 2, although they were used in the form of Fig. 3 for computing.

4. <u>The group-velocity.</u>

In order to obtain the path length, it will be necessary to integrate 1/U. But at the reflection U = 0 and 1/U becomes infinite. Moreover the last portion of the path near U = 0 will evidently be of great importance as the group spends a long time in the part of the ionosphere where its velocity is small, so the graphical representation must be arranged to be accurate near $\delta = 0$.

Now when $\tau = 0$, $U = \sqrt{\delta}$, so that $U/\sqrt{\delta} = 1$.

When $\tau \neq 0$, and δ is small $U \to \sqrt{\delta}\cos D$, i.e. $\frac{U}{\sqrt{\delta}} \to \cos D$. and

it is seen that $U/\sqrt{\delta}$ which is the ratio of velocities in the two cases is everywhere finite. It is therefore a convenient quantity to plot especially as it turns out later to be specially useful when integrating.

The limit-value for $\tau \to \infty$ is, in a suitable form for computing:-

$$\frac{U}{\sqrt{\delta}} \longrightarrow \frac{(1 - \zeta\sin^2 D)^{1/2}}{1 - Z^2\zeta\sin^2 D}$$

Curves of $\frac{U}{\sqrt{\delta}}$ as functions of δ are given in Fig. 4 although as before computations are made from a set showing $U/\sqrt{\delta}$ as a function of f.

5. <u>The attenuation.</u>

If $\tau = 0$ $\quad k = 1/\sqrt{\delta}$

Thus the effect of the field is expressed by $k\sqrt{\delta}$ which remains everywhere finite so that as in the case of the group-velocity it is better to plot a derived quantity rather than the absorption directly. $k\sqrt{\delta}$ will also be found to have the same advantage for integrating.

At $\zeta = 1$ $\quad k\sqrt{\delta} \longrightarrow \sec D$

At $\tau = \infty$ putting in the limiting values already given shows that

$$k \longrightarrow \frac{\cos^2 D}{(1 - \zeta)^{1/2}(1 - \zeta\sin^2 D)^{3/2}}$$

The case of ζ small $\left(\text{i.e. } \lim_{\zeta \to 0} k\right)$ is of particular importance as it expresses the "limiting attenuation" of a non-deviating layer such as is known to be of great importance, especially in the daytime. This limiting value is

$$k\sqrt{\delta} = k_0 \longrightarrow p + \frac{2F_xF_w}{\tau^2}$$

For $D = -67^\circ$, curves of k_o as functions of δ are given in Fig. 5.

On account of the importance of the limiting attenuation, it has been computed as a function of τ and D for the whole range of D and is shown in Fig. 6. It has been customary in this region to make use of the "quasi-longitudinal" approximation which is equivalent to

$$k_o = \frac{1}{(1 + \tau \sin D)^2}$$

and it is evident that the approximation is correct for D = 0 and also for $D = 90^\circ$, but in between gives too high a value. The ratio of the two expressions is shown in Fig. 7 for several values of D. It is to be noticed that although this work has so far been presented in terms of vertical incidence, this restriction can be removed for the non-deviative part, and the expression found can be quite generally used taking D as the angle between the field and the wave-front. For oblique incidence it is only necessary to include a geometrical factor to express the increase in path-length in the non-deviative region.

6. General properties of the differentials.

Before proceeding to integrate the differentials and obtain the observable quantities, it is best to see what information can be obtained without integrating.

It is evident from the curves that over most of the range of δ, the group-velocity is greater than it would be without the field and the absorption is less. This is what is to be expected physically, as the electrons are moved by the wave against the direction which the field would give them, and their motion is therefore reduced by the field. But near the reflection point these conditions are reversed, the group-velocity decreases and the absorption increases. This is already shown in the curves given by Bajpai and Mathur[3] and is inherent in the expressions given by Goubau[4], but in both cases is passed by without comment, although for practical purposes it is extremely important. The physical reason for it is seen from the values of $\frac{du}{dz}$. The electron motion is tilted by the field out of the plane of the wave-front, with the result that the wave-packet moves sideways.

The actual group-velocity is not cU but is given by $cU \sec\left(\tan^{-1}\frac{du}{dz}\right)$. This expression is always greater than $\sqrt{\delta}$, and becomes equal to $\sqrt{\delta}$ only at the reflection point where the group is travelling perpendicularly to the field and, being an ordinary wave, is not affected by the field. Similarly the absorption for unit distance travelled is always less than without the field, the increase near the reflection is produced by referring it to the vertical component of the path.
The fact that the path has to become perpendicular to the field at the reflection point provides a physical explanation of the independence of the critical frequency from the field. The ordinary wave is in fact reflected in the condition in which its propagation does not depend on the field.

The appreciation of these facts shows that it is not possible to predict from the differentials in what way the integrals will be affected by the field. The differentials must be integrated with respect to vertical height, and in the case for example, of the path-time, will be less over part of the path, and greater over the remainder. The second part will normally be short because it is only where δ is small, but as the velocity is also small, the time taken is appreciable. It will be shown that in fact in typical cases both the path-time and the absorption are increased, but there may be occasions when they are decreased. Differing density distributions completely alter the relative importance of the two parts of the path.

PART II. INTEGRALS.

1. Methods.

To obtain observable quantities it is necessary to evaluate the integrals

$$P' = \int_0^{z_0} \frac{1}{U}\,dz$$

$$Q = \int_0^{z_0} \frac{du}{dz}\,dz$$

$$K = \frac{2\pi}{\lambda}\int_0^{z_0} \frac{\alpha}{2}\,\zeta k\,dz$$

for given relations between the density N, the collision frequency f_c as expressed in δ and α, and z , z_0 being the value of z at which $\delta = 1$. Strictly, this gives only half the path, but as the upgoing and downcoming branches are symmetrical at vertical incidence, this half is sufficient. The truly observable quantities are 2P' and 2K. The overall value of Q on reflection is zero, but it is included as the value at the top of the path is the total sideways deviation, and measures the extent over which the condition of horizontal stratification must be satisfied for the theory to be strictly applicable.

All the integrals may be expressed in the general form

$$I = \int_0^{z_0} L\,dz$$

where L is known as a function of ζ and in the integrals for P' and K, becomes infinite at $z = z_0$. If the distribution of density is known, this gives the relation of ζ to Z, and the integral becomes

$$I = \int_0^1 L\,\frac{dz}{d\zeta}\,d\zeta$$

Since L is only known graphically, the integral must be obtained by a numerical method, and difficulty arises where L becomes infinite. There are known ways of dealing with this (as suggested for example, by Booker)[5], but in the general case it is necessary to fit an analytic function to L in some way. Since it has been seen that an exact determination of this part of the integral is of great importance, it is better to obtain analytic transformations which avoid the infinity in the integrand. The method used is essentially one of those given by Millington[1], and is recapitulated here. In order to obtain numerical values which will show the magnitude of the effect to be expected a "parabolic" ionosphere is chosen in which

$$1 - \frac{N}{N_m} = \left(1 - \frac{z}{z_m}\right)^2$$

where the maximum density N_m occurs at the height z_m. (Millington also deals with the "sine-squared" region).

Now if f_{om} is the value of f_o corresponding to N_m and

$$f^2/f^2_{om} = 1/\zeta_m = x^2$$

$$\frac{N}{N_m} = \frac{f_o^2}{f_{om}^2} = \frac{f_o^2}{f^2}\,\frac{f^2}{f_{om}^2} = \zeta/\zeta_m$$

Thus $1 - \zeta/\zeta_m = \left(1 - \frac{z}{z_m}\right)^2$

and $\frac{dz}{d\zeta} = \frac{z_m}{2\zeta_m}\;\frac{1}{\sqrt{1 - \zeta/\zeta_m}}$

Further substitutions made are

$$\zeta_m = 1 + \ell$$

so that $\ell = 0$ at the penetration frequency.

$$\zeta = 1 - \gamma^2$$

so that γ goes from 1 to 0 as ζ goes from 0 to 1.

It is also to be noticed that $\gamma = \sqrt{\delta}$

$$\xi = \frac{\log\left(1 + \frac{\gamma}{\sqrt{\ell}}\right)}{\log\left(1 + \frac{1}{\sqrt{\ell}}\right)}$$

and ξ goes from 0 to 1 as γ goes from 1 to 0.

Then the integral becomes

$$I = z_m C \int_o^1 F \gamma L \, d\xi$$

where $C = \frac{1}{\sqrt{1+\ell}} \log\left(1 + \frac{1}{\sqrt{\ell}}\right)$

$$F = \frac{\sqrt{\ell} + \gamma}{\sqrt{\ell} + \gamma^2}$$

The only infinity is then in C at $\ell = 0$, the escape frequency, because the order of infinity in L in these problems is such that $F \gamma L$ is finite. When integrating through a particular layer, ℓ is determined by the frequency, and γ by δ. Curves have been drawn from which ξ and F are easily found.

In the work that follows interest is centred not so much on the actual value of the integral as in the effect of the magnetic field. If the field is not present, the integrals can be calculated analytically, and the figures presented are the ratio of the two cases. Before proceeding to the numerical integrations, some limiting values which can be found analytically are given.

2. End-points of the integrals.

1. $x = 0$ (or $\tau = \infty$)

At $x = 0$ (zero frequency) ℓ becomes infinite and C zero, but F, γ L are finite. Thus the integral has a definite value, and since C does not depend on the field, the ratio of the two cases remains finite.

The ratio is more easily obtained by going back to the original integral

$$I = \int_0^1 L \frac{dz}{d\zeta} \, d\zeta$$

$$= \frac{z_m x^2}{2} \int_0^1 \frac{L}{\sqrt{1 - x^2}} \, d\zeta$$

As x tends to zero the integral tends to $\int_0^1 L \, d\zeta$ and using L_o for $\tau = 0$ and L_H for $\tau = \infty$ the ratio becomes

$$R = \frac{\int_0^1 L_H \, d\zeta}{\int_0^1 L_o \, d\zeta}$$

and as the values of L_H for $\tau = \infty$ have been expressed analytically R can be calculated and acts as a check on the remainder of the work.

(a) Path-length.

In the case of path-length, $L = \frac{1}{U}$.

For $\tau = 0$, $U = \sqrt{\zeta}$

and $\int_0^1 L_o \, d\zeta = \int_0^1 \frac{d}{\sqrt{1 - \zeta}} = 2.$

When $\tau = 0$, it has already been shown that

$$\frac{1}{U} = \frac{1 - Z^2 \sin^2 D}{(1 - \zeta \sin^2 D)^{1/2} (1 - \zeta)^{1/2}}$$

in which $Z^2 = \dfrac{1 - \zeta}{1 - \zeta \sin^2 D}$

Thus $\displaystyle\int \frac{1}{U} \zeta \, d = \int_0^1 \frac{1 - 2\zeta \sin^2 D + \zeta^2 \sin^2 D}{(1 - \zeta \sin^2 D)^{3/2} (1 - \zeta)^{1/2}} \, d\zeta$

This integral is easily evaluated, and the final result is

$$R_p = \frac{\int_0^1 L_H \, d\zeta}{\int_0^1 L_o \, d} = \frac{3}{2 \sin^2 D} \left[1 - \frac{\cos^2 D}{\sin D} \tanh^{-1} \sin D \right]$$

For $D = 67^\circ$, the case for which all the other work is done, this gives

$$R_p = 1.316$$

This value could equally well be found by integrating the values of $\frac{U}{\sqrt{\zeta}}$ already given by means of the graphical method. This was done as a check on the overall accuracy and gave $R = 1.30$ which is taken to be sufficiently accurate.

As an example of the way in which the magnetic field effect depends on D, it may be noted that according to this equation, $R = 1$ for $D = 0$ at the magnetic equator, agreeing with the known independence of the ordinary from the field at the equator, and R takes its maximum value of 1.5 at $D = \pm 90$, i.e. at the magnetic poles. Here, however the theory may break down on account of the coupling with the isolated branch of the $Z - \zeta$ curve.

(b) The Attenuation.

The limiting value of the attenuation at $\tau = \infty$ can be similarly dealt with. When $\tau = 0$ $L = \zeta / \sqrt{1 - \zeta}$ on the assumption that the collision frequency is constant

Thus $\displaystyle\int_0^1 L_o \, d\zeta = \int_0^1 \frac{\zeta}{\sqrt{1 - \zeta}} \, d\zeta = \frac{4}{3}$

When $\tau \neq 0$ as already shown

$$L = \zeta \quad k \rightarrow \frac{\zeta \cos^2 D}{(1 - \zeta)^{1/2} (1 - \zeta \sin D)^{3/2}}$$

Evaluation of the integral gives

$$R_a = \frac{\int_0^1 L_H \, d\zeta}{\int_0^1 L_o \, d\zeta} = \frac{3}{\sin^2 D} \left[1 - \frac{\cos^2 D}{\sin D} \tanh^{-1} \sin D \right]$$

This is the same expression as has already been found for the path length, and thus for $D = 67^{\circ}$, $R_a = 1.316$.

(c) Lateral deviation.

In this case there is no deviation at all when $\tau = 0$, and consequently the ratio would have no meaning. The effect of the field is expressed entirely by the actual value of deviation, which tends to zero at $\tau \longrightarrow \infty$ because there is no penetration of the layer (unless the critical frequency is also very low.)

3. Conditions at x = 1, the critical frequency.

As x approaches unity, ℓ becomes small. The integral is still given formally by

$$I = z_m C \int_0^1 F \gamma L . \, d\zeta$$

and the infinity which occurs in path-length and absorption is only in C so that the integral is finite. F becomes equal to unity, but there is some apparent difficulty in finding ζ as $\zeta \longrightarrow 1$ independently of γ except actually at $\gamma = 0$ where $\zeta = 0$. This means that if γL is not zero at $\zeta = 0$, the integral becomes

$$\lim_{\ell \longrightarrow 0} \frac{I}{m^C} = \int_0^1 \lim_{\gamma \to 0} (\gamma L) \, d\zeta = \lim_{\gamma \to 0} (\gamma L).$$

Physically this means that where L has an infinity at the reflection point $\gamma = 0$, the wave-packet spends an increasingly large proportion of its total time near the reflection point as $x \longrightarrow 1$, until at the critical frequency the integral is determined only by the conditions at the reflection point.

Since C does not depend on the field, the ratio which expresses the effect of the field becomes

$$\lim_{\ell \longrightarrow 0} S = \frac{\lim_{\gamma \to 0} \left(\gamma L_H\right)}{\lim_{\gamma \to 0} \left(\gamma L_o\right)}$$

(a) Path-length. $L = 1/U$.

At $\gamma = 0$ which is $\delta = 0$, $L_o = 1/\sqrt{\delta}$

$$L_H \longrightarrow 1/\sqrt{\delta} \cos D$$

Thus $S_p = \sec D$.

(b) Attenuation. $L = \zeta k$.

At $= 0$ $L_o = \zeta/\sqrt{\delta}$

$$L_H \longrightarrow \zeta/\sqrt{\delta} \cos D.$$

and $S_a = \sec D$.

Both attenuation and path-length are increased by the field in the ratio sec D at the critical frequency. For D = 67 sec D = 2.559 so that the effect is considerable. Furthermore, it is now seen that both quantities are increased by the field at each end of the h'f curve, and this leads to the suspicion that they will be increased on all frequencies. The later detailed integrations will show that for a parabolic layer at least, this suspicion is confirmed. It is also found that the increase, although considerable at all frequencies runs up to its limiting value very steeply as the critical frequency is approached.

(c) Lateral deviation.

The method described for $x \to 1$ does not apply for the lateral deviation as $L = \frac{du}{dz}$ does not become infinite. Thus $\gamma L \to 0$ and the resulting zero together with the infinity on C gives a finite value. But as L is not infinite, there is no need to use the complete transformation and deviation may be obtained directly from

$$Q = \int_0^z \frac{du}{dz}\, dz$$

for at the critical frequency x = 1 and

$$1 - \zeta = \left(1 - \frac{z}{z_m}\right)^2$$

Thus $\frac{z}{z_m} = 1 - \sqrt{\delta}$ (in the lower half of the layer, which is the interesting part)

and $dz = z_m d\,(1 - \sqrt{\delta})$,

from which $Q_m = z_m \int_0^1 \frac{du}{dz}\, d\,(\sqrt{\delta})$

and the limiting value of Q is found graphically by plotting $\frac{du}{dz}$ against $\sqrt{\delta}$

The value so found evidently depends on the way in which $\frac{du}{dz}$ varies through the layer, and there is no simple analytical expression for its limiting value. However, it is evident that for a given value of the critical frequency f_{om}, it is possible to write

$$Q_m = z_m\ M\,(f_{om})$$

As f_{om} increases, the effect of the field decreases so that the integral is only appreciable over a diminishing range of $1 - \sqrt{\delta}$ and M tends to zero. It is of interest to calculate the limit-value as f_{om} becomes small. It has been shown that at zero frequency

$$\frac{du}{dz} = \frac{-\zeta \cos D \sin D}{1 - \zeta \sin^2 D}$$

$$\text{So thus } (M)_{max} = \int_0^1 \frac{\zeta \cos D \sin D}{1 - \zeta \sin^2 D}\, d\,(\sqrt{\delta})$$

$$= D \operatorname{cosec}^2 D - \cot D.$$

For $D = 0$ $(M)_{max} = 0$

$D = 90^\circ$ $(M)_{max} = \pi/2.$

$D = 67^\circ$ $(M)_{max} = 0.956.$

Graphical integration for this last value gave 0.962, again giving a good check on the numerical accuracy.

4. Numerical results.

In order to obtain representative numerical results the integrations have been carried out through three parabolic layers having critical frequencies of 2 Mc/s, 5 Mc/s and 10 Mc/s and compared with the no-field values.

(a) Path-length.

The path-length is given by

$$P' = \int \frac{1}{U}\, dz = z_m\, 0 \int F \frac{Y}{U}\, d$$

But $Y = \sqrt{\delta}$ and $U/\sqrt{\delta}$ is already known so that the integrand is easily found. The end-points are known, for at $\zeta = 0$ $F = 1$ and $\frac{Y}{U} = \sec D$ whereas at $\zeta = 1$ F has a value depending only on ℓ and $Y/U = 1$

A number of the curves obtained when integrating through the 2 mc/s layer are shown in Fig. 8. For zero field $Y/U = 1$ and the integrand is F. This is also shown, and integrating F leads to the well known expression

$$h'_o = x\, z_m \tanh^{-1} x.$$

The curves of Fig. 8 show how the overall effect of the field is reduced by the compensation of two large effects in opposite directions. Over much of the range of ζ towards $\zeta = 1$ the curve lies below the no-field line, and the reduction in time, measured by the area between the curve and the line, is considerable. But over the last part of the range of towards $\zeta = 0$, the curve is above the line, and the increase in time over this part is also considerable. The final result is the small difference of these two large areas, which in the case of the parabolic layer, very nearly compensate.

Using h'_H for the apparent heights deduced from Fig. 8, the values of h'_H/h'_o as a function of x for the three layers are shown in Fig. 9. It is evident that the increase in h' is considerable. At $x = 0$ h'_H/h'_o is 1.316 as already calculated, but the rise to sec D which must occur is confined to values of x very close to unity. However, the simple theory shows that values of x greater than 0.99 are rarely of great importance as they are usually lost by absorption, so the detailed way in which h'_H/h'_o approaches sec D has not been investigated. The observable effect may also be represented as in Fig. 10 which shows $(h'_H - h'_o)\, z_m$ as a function of x. A representative value of z_m for region F is 100 km. so that it is seen that the increases in height are easily measurable.

(b) Attenuation.

The attenuation integral is

$$K = \frac{2\pi}{\lambda} \int \frac{\alpha}{2} \xi k \, dz = \frac{2\pi}{\lambda} \int \frac{\alpha}{2} \xi F. \quad k\sqrt{\delta} \quad d\xi$$

For calculation, the assumption has been made that the collision frequency is constant throughout the layer. For $\tau = 0$ this leads to

$$K = \frac{zm}{2}\left[-1 + \frac{1 + x^2}{x} \tanh^{-1} x\right]$$

The values of the ratio K_H/K_o are shown in Fig. 11. The curves have the same general shape as those for h'_H/h'_o from 1.316 at x = 0 and a very steep rise to sec D near x = 1.

In this case again, the increase is quite measurable but in agreement with the results of Millington, the relation of h' to K near the critical frequency is the same as if the field were not present, as both are altered in the same manner.

(c) Lateral deviation.

It is strictly unnecessary to transform the integral for lateral deviation into the ξ form, as there is no infinity in $\frac{du}{dz}$. It was however, convenient to do so as the relevant graphs were already available. The total sideways deviation at the reflection point is

$$Q(x) = z_m q$$

and curves are given of q in Fig. 12. Again the deviation is only considerable near x = 1. In Fig. 13 is shown the value of q for x = 1 as function of the critical frequency. This is the maximum deviation that can occur in a layer of this type.

PART III - THE EFFECT OF THE LORENTZ TERM

It would not be difficult in principle to compute the effect of the Lorentz term from the complete expressions. Their added complexity would add considerably to the labour, and therefore only an estimate of the effect to be expected has been made. By approximate methods it is, however, possible to see that if due allowance is made for the known effect of the Lorentz term in the no-field case, it is not likely that its inclusion in the more complex treatment will add much of importance on the ordinary ray.

1. The effect without the field.

It is convenient firstly to recall the effect produced in the absence of the field. This has been fully worked out by Ratcliffe who obtains for a parabolic layer of the type already used

$$\frac{h'_L}{Z_m} = \sqrt{2}\,(1 + 2x^2)^{-\frac{1}{2}} \left[(1 + 4x^2)\; E(\alpha_0\theta) - (1 - 2x^2)\; F(\alpha_0\theta)\right]_{\theta = \theta_0}^{\theta = \pi/2} - \frac{2}{3} k^2$$

in which h'_L = equivalent height including the Lorentz term

$$k^2 = \sin^2 \alpha_0 = 3x^2/(1 + 2x^2)$$

$$\theta_0 = \sin^{-1}\sqrt{\frac{2}{3}}$$

and E, F are the elliptic integrals of the second and first kinds. The relation between penetration frequency and density is altered, but for comparison purposes use is made of regions having the same penetration frequency. Thus the corresponding expression for the equivalent height h'_S without the Lorentz term is

$$h'_S = z_m\, x \tanh^{-1} x.$$

The complete curve of h'_L against x has been drawn by Ratcliffe, but it is more instructive to consider the ratio h'_L/h'_S.

At low frequencies, i.e. x small and

$$k^2 = 3x^2 = \sin^2 \alpha_0 = (\alpha_0)^2$$

so that k and α are also small.

Expanding the elliptic integrals in terms of x gives, after reduction

$$\left(h'_L\right)_{x \to 0} = z_m \left[\frac{9\sqrt{2}}{2}\left(\frac{\pi}{2} - \theta_0\right) - 3\right] x^2$$

Similarly $\left(h'_S\right)_{x \to 0} = z_m x^2$

Thus $\left(\dfrac{h'_S}{h'_L}\right)_{x \to 0} = 1.092.$

Near the penetration frequency x approaches unity and h'_L becomes infinite because of the infinity in $F(\alpha_0\; \pi/2)$

Putting $k' = \cos \alpha_0$

and expanding in terms of k' gives

$$F(\alpha_0\, \pi/2) \quad \log 4/k'$$

and $(h'_L)_{x \to 1} = z_m \frac{1}{2}\sqrt{\frac{2}{3}} \quad \log \frac{24}{1 - x}$

Similarly

$$(h'_S)_{x \to 1} = z_m\, \frac{1}{2} \log \frac{2}{1 - x}$$

and thus $\left.\frac{h'_s}{h'_L}\right|_{x \to 1} = \sqrt{\frac{3}{2}} = 1.226$

Further computation confirms that h'_S is in fact always greater than h'_L, and that as x tends to unity the difference increases to infinity in such a way that the ratio is $\sqrt{3/2}$. This shows that the curve of h'_L will be indistinguishable from that of h'_S with a value z'_m where.

$$z'_m = a z_m$$

neglecting the fact that the curve is not strictly parabolic at lower values of x.

If use is made only of the portion very near x = 1, a = 1.226, but if a larger portion of the curve is used, a will be slightly less. This equivalence has already been noted by Beynon[7].

Approximation with magnetic field.

In making use of the expressions, a slight modification is needed in the transformation of the integral. The quantity L to be integrated is a function of η and the integral becomes

$$I = \int L dz = \int L \frac{dz}{d\zeta} \frac{d\zeta}{d} d\eta$$

$$\text{Now } \frac{dz}{d} = \frac{z_m}{2\zeta_m} \frac{1}{\sqrt{1 - \zeta/\zeta_m}}$$

$$= \frac{z_m}{2\eta_m} (1 - \ell\eta_m) \sqrt{\frac{1 - \ell\eta_m}{1 - \eta/\eta_m}}$$

$$\text{and } \frac{d\zeta}{d\eta} = 1/(1 - \ell\eta)^2$$

so that

$$I = \frac{z_m(1 - \ell\eta_m)}{2\eta_m} \int L \frac{1}{(1 - \ell\eta)^{3/2}} \frac{1}{(1 - \eta/\eta_m)^{\frac{1}{2}}} d\eta$$

The transformations previously used were designed to deal with the infinities in the first and third terms of the integral. These are exactly the same in this case, and the extra term does not contribute another infinity, so that the same transformations are appropriate, and the integral becomes

$$I = z_m C (1 - \ell\eta_m) \int_0^1 \frac{L}{(1 - \ell\eta)^{3/2}} \gamma F \, d\zeta$$

The end-points are found in the same way as before. At zero frequency m tends to $1/\ell$, and the ratio of the two cases is given by the ratio of the integrals,

$$I_o = \int_o^1 \frac{L}{(1 - \ell\eta)^2} \, d\eta$$

Near the penetration frequency for the ordinary, η_m approaches unity and the integral breaks down in the same way as before, and once again the ratio which expresses the effect of the field becomes

$$\lim S = \frac{\lim_{\gamma \to 0} (\gamma L_H)}{\lim_{\gamma \to 0} (\gamma L_o)}$$

and in this case $\gamma = \sqrt{\varepsilon} = \sqrt{1 - \eta}$.

The limiting values near the penetration are

$$\frac{du}{dz} \longrightarrow - \tan D$$

$$\frac{dP^1}{dz} \longrightarrow \frac{(1 - \ell) \sec D}{\sqrt{\varepsilon}} \quad \text{or} \quad \frac{1 - \ell}{\sqrt{\varepsilon}}$$

$$K \longrightarrow \frac{(1 - \ell) \sec D}{\sqrt{\varepsilon}} \quad \text{or} \quad \frac{1 - \ell}{\sqrt{\varepsilon}}$$

Thus the limiting value of the tilt is the same as before, and the ratio of the integrals for path-length and attenuation also remains equal to sec D independently of the Lorentz term.

Consequently, although this is not a complete proof, it is reasonable to assume that the effect of the field at other frequencies than the penetration will not depend markedly on the value of the Lorentz term, so long as attention is confined to the ordinary ray. For this reason the arithmetic has not been undertaken in detail.

PART IV CONCLUSIONS.

Although only a few h'f curves have been calculated in detail, it is possible to draw some general conclusions.

The sideways tilt is of importance, and it is shown that for quite normal values of the critical frequency the layer must be only horizontally stratified for at least 50 Kms. in the magnetic meridian for the "critical frequency" found in the ordinary way to have strictly the meaning usually given to it. The extraordinary is deviated in the opposite direction to an extent not calculated but almost certainly considerably less, so that the ionospheres to which the two critical frequencies refer are probably often separated by a distance of the order of 70 Kms. The magnitude of the gradient may be estimated from the difference between critical frequencies at Burghead and Baddow which lie roughly N - S. Even on the average over a month this is often of the order of 0.5 Mc/s. As the distance is about 700 Kms. the difference in critical frequencies for the two rays can average as much as 0.05 Mc/s. an easily measurable amount. On individual days the gradient may be considerably more, so that this lateral deviation is in itself enough to account for many of the variations in separation between critical frequencies which are actually observed, although it is not suggested that this is the only possible reason. The fact that near the reflection point, where its effective wave-length is fairly long, the packet is travelling almost perpendicular to the field, will also be of importance in connection with irregularities which may be stratified in that direction.

The effect on the calculation of oblique transmission cannot be directly seen. In order to secure certainty it would be necessary to work out the detailed oblique theory in the same way as the vertical theory is treated here. This has not yet been done because of the magnitude of the labour involved, but it is hoped to return to the oblique problem later. Meanwhile it is possible to make some reasoned guesses at the behaviour to be expected at oblique incidence, by using the theory in conjunction with the known experimental values.

It is known that the simple theory neglecting the field gives extremely accurate agreement with the observed values, at fairly high angles of elevation, but appears to give rather low values for skip frequencies at long distances. This presumably means that the effect of the field at oblique incidence is of the same order as at vertical incidence and that the simple theory gives the right value for the wrong reason. The curves of Fig. 8 show also how the magnitude of the field-effect is given by a rather delicate balance between two large and opposite effects which may be easily upset. It is because of this balance that considerable numerical accuracy is necessary in the $Z - \zeta$ curves. Thus if instead of a parabolic layer, a layer with a "shelf" on the underside were used, the portion of the curve below the no-field line would be extended, which might make $h'_H < h'_o$ over parts of the h'f curve, although near the critical frequency the same limit must be approached as before. A similar balance is to be expected at oblique incidence, and it cannot be assumed that it will be maintained to low angles. In fact as the frequency to be used at low angles will be much higher, it is to be expected that the magnetic field effects will be reduced. Since $h'_H > h'_o$ this would in fact give an increase in the skip-frequency, a result which is at least in the right direction.

A direct evaluation can be made of the effect on the usual means of skip-calculation. The curvature of the relation of h'_H/h'_o to x shown in Fig. 9 means that it will not in general be possible to fit a parabolic region to the observed h'f curve, but a plot of h'_H against h'_o is found to be linear between $x = 0.7$ and $x = 0.97$, within the limits of observation. The lines are shown in Fig. 14. Using H_h and H_o for the heights of maximum density deduced on the two theories and z_{mh}, z_{mo} for the semi-thickness, the curves give

Critical frequency.	2 mc/s.	5 mc/s.	10 mc/s.
z_{mh} / z_{mo}	1.45	1.32	1.22
$\frac{H_H - H_o}{z_{mo}}$	0.25	0.13	0.007

Thus the observed effects are considerable, and emphasize the fact that the parameters derived in fitting a parabolic layer to an observed h'f curve are not to be taken as necessarily physical quantities. It also means that h'f curve is modified in such a way that the portion near the critical frequency behaves roughly like the h'f curve of a different parabolic layer with a greater height of maximum density and considerably greater semi-thickness.

It is known that although it is occasionally possible to fit a parabolic layer to a large portion of the h'f curve by means of the simple theory, the values of h' at the lower frequencies are generally considerably greater than would agree with the parabola deduced from the cusp. This is the type of effect that would be produced by the field, so that the true distribution may be more nearly parabolic than the simple theory would allow one to believe.

The number usually employed to determine oblique transmission is the skip-factor for 2,500 Kms. This depends on the height of the layer above the ground and taking values of this height of maximum density h_m between 200 and 400 Kms. and a true z_m of 80 Kms. the factor F_H deduced from the actual h'f curve and the factor F_o deduced by applying the simple theory to the actual layer can be compared. F_H is always less than F_o and Fig. 15 shows $F_o - F_H$ plotted against F_o (or h_m). The change in the factor is evidently appreciable and it is remarkable that if the extrapolation to long-distance from F_H is made on the basis of F_o, the correction is of the order which has been assumed to fit the observed long-distance transmission.

The effect on the deviative attenuation is of less importance as less detailed work has been done on the simple theory. The results suggest that the overall attenuation is increased at all frequencies, and not only at the critical frequency. The detail in practice will evidently be greatly affected by the precise variation of collision frequency with height and the numerical values given cannot be too closely compared with practical values. The work has also led to a detailed survey of the non-deviative attenuation which is of considerable practical importance, and shows the extent to which the "quasi-longitudinal" approximation may be misleading.

PART V. SUMMARY

The important results found for the ordinary ray are:-

(1) The apparent height and attenuation are increased by the magnetic field

(2) This increase is easily measurable and makes an appreciable difference to the "skip-factor".

(3) The overall path-length is the result of a delicate balance between opposing effects, easily upset by changing the density-distribution or angle of incidence.

(4) The non-deviative attenuation is not correctly represented by the quasi-longitudinal approximation. True values are given.

(5) The condition of "horizontal" stratification must be maintained over at least 70 Kms. in the meridian for a true "critical frequency" to be obtained.

(6) The inclusion of the "Lorentz term" would not make much observable difference.

Acknowledgements.

This work has been greatly facilitated by both Mr. Eckersley and Mr. Millington, both by their permission to use their analysis and by their interest in its progress. Special thanks are also due to Miss E. Andrew, W.R.N.S., who has undertaken nearly all of the detailed arithmetic involved.

References.

1. G. Millington, Proc. Phys. Soc. 50 561 (1938).

2. G. Millington.

3. Bajpai and Mathur, Indian Journ. Phys. 11 165 (1937).

4. G. Goubau, H.F.T. 44 17 (1934).

5. Booker, Phil. Trans. 237 411 (1938)

6. Ratcliffe, Proc. Phys. Soc. 51 747 (1939)

7. Beynon. R.R.B. paper C/82.

JWC/JOB.

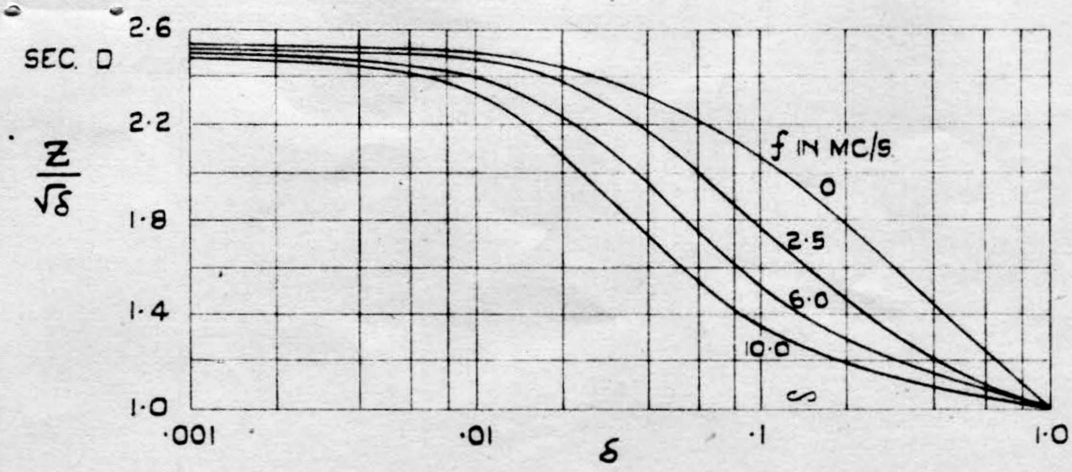


FIG. 1.

REFRACTIVE INDEX.

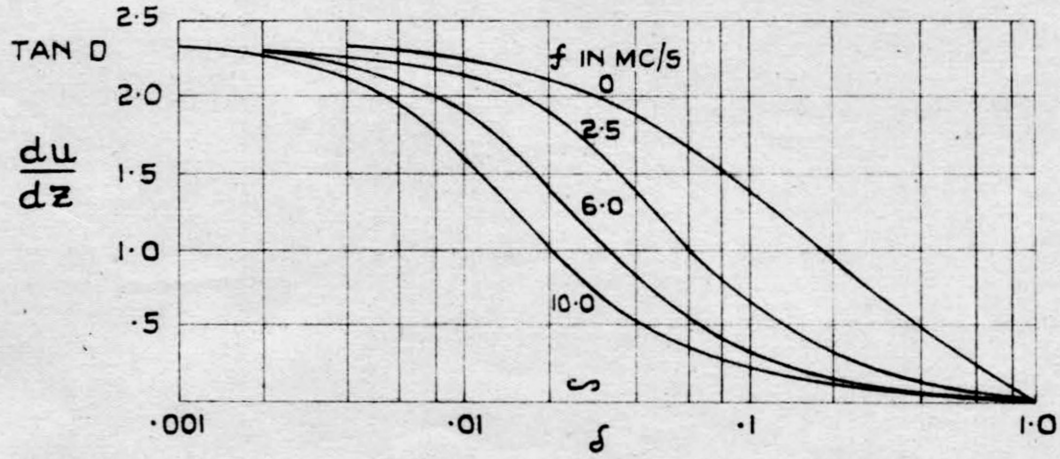


FIG. 2.

LATERAL DEVIATION.

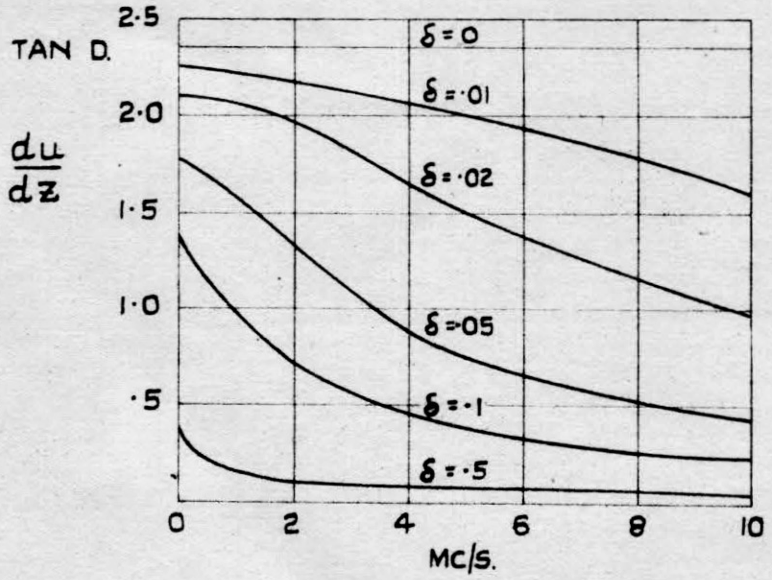


FIG. 3.

LATERAL DEVIATION.

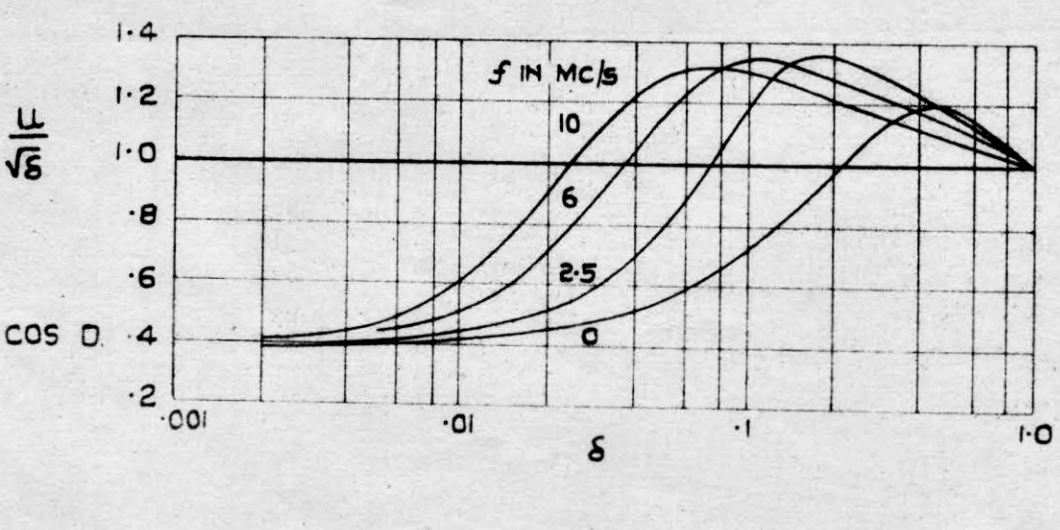


FIG. 4.

GROUP VELOCITY

RD/SK 4450

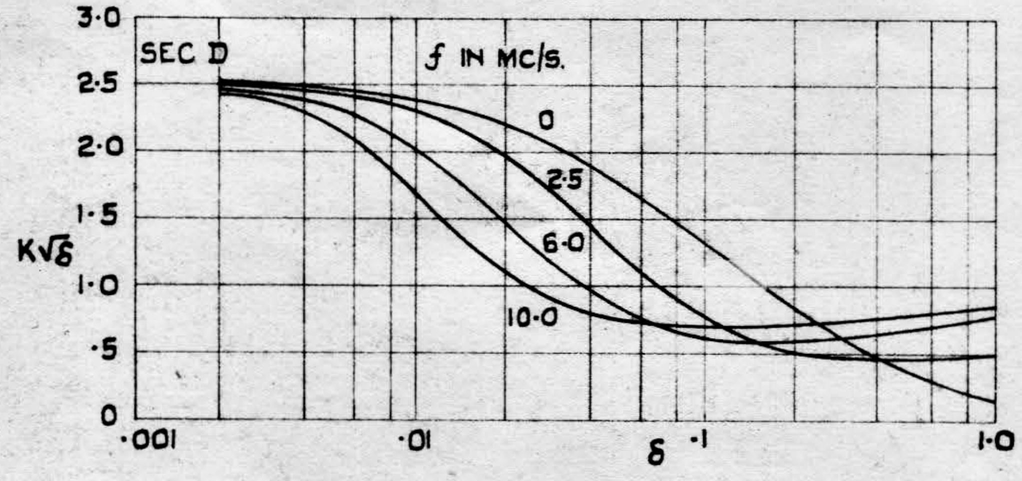


FIG. 5.

ATTENUATION

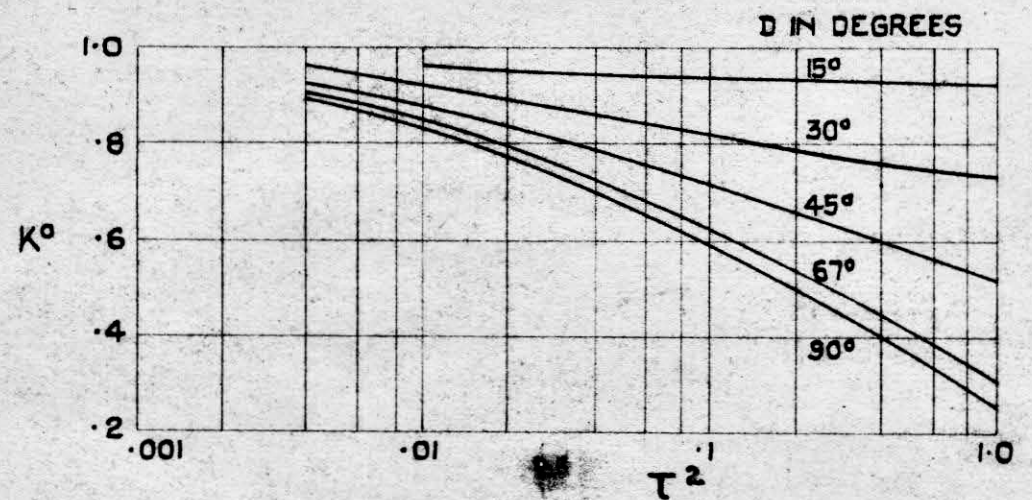
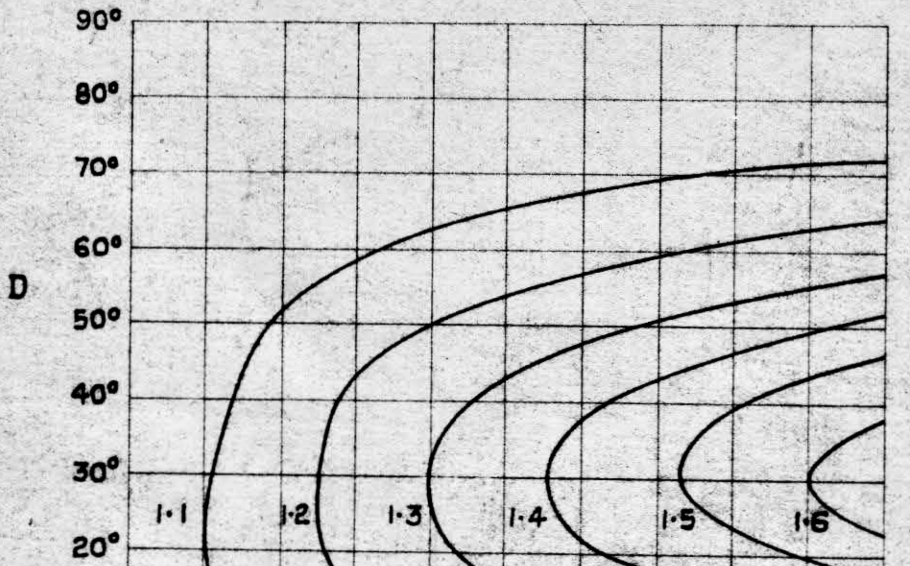


FIG. 6.

LIMITING ATTENUATION.
(ORDINARY RAYS.)



CURVES OF VARIOUS VALUES OF RATIO.

FIG. 7.

ERRORS OF "QUASI-LONGITUDINAL" APPROXIMATION.

RD/SK 4451

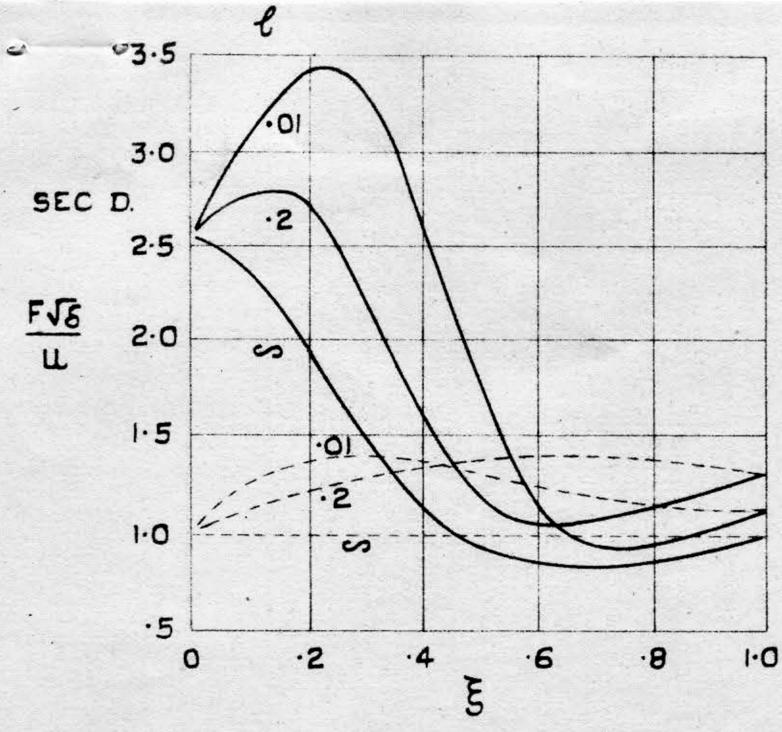


FIG. 8.

INTEGRALS FOR PATH IN 2 MC/S LAYER.

——— WITH FIELD.
- - - - - WITHOUT FIELD.

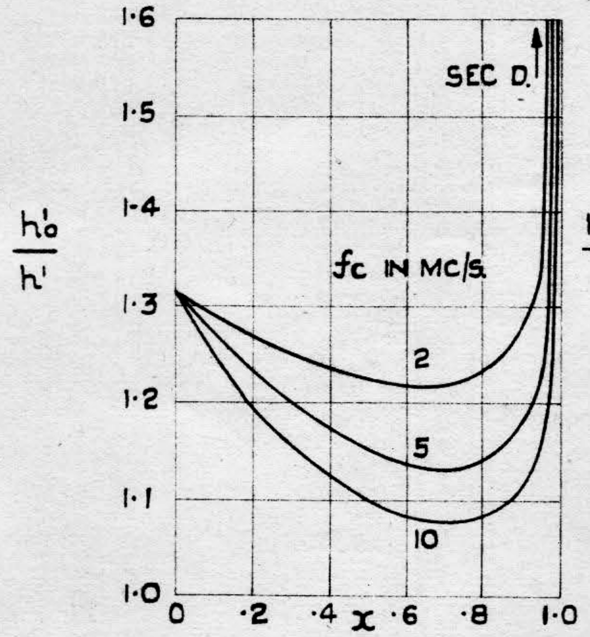


FIG. 9

RATIO OF $\frac{h'_0}{h'}$

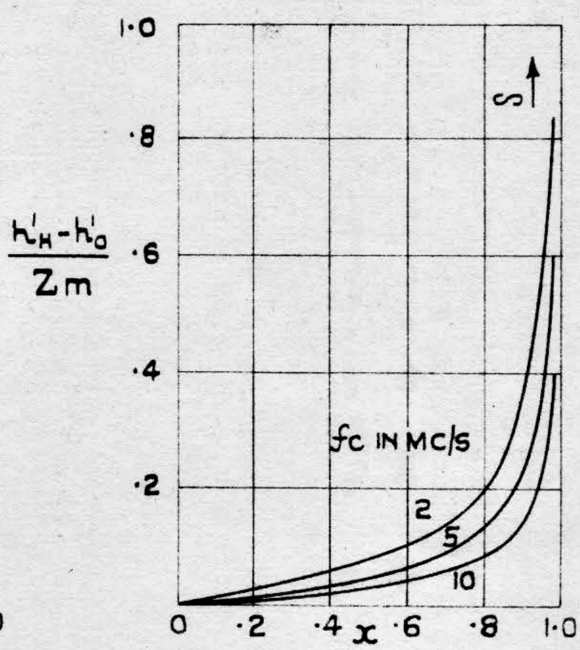


FIG. 10.

INCREASE OF h'

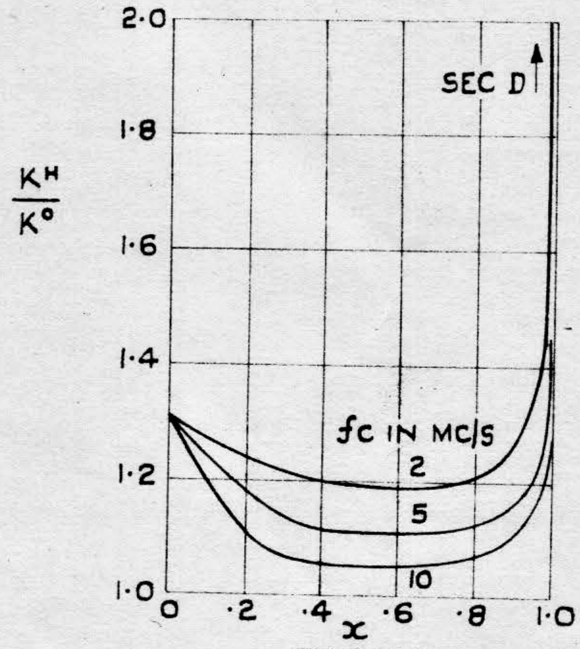


FIG. 11.

ATTENUATION.

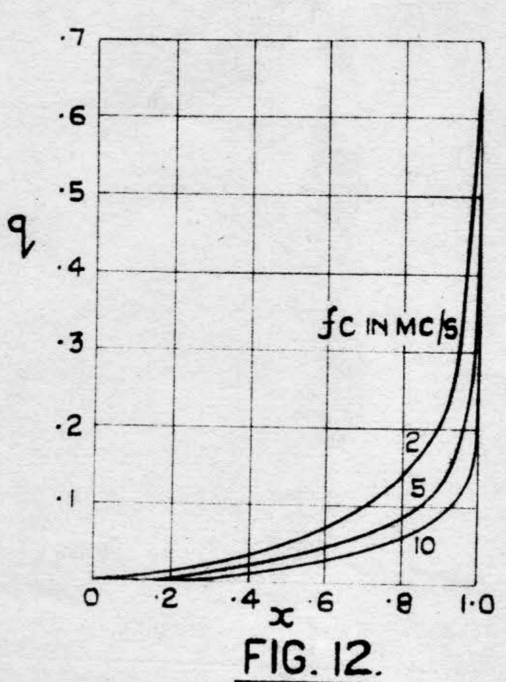


FIG. 12.

LATERAL DEVIATION

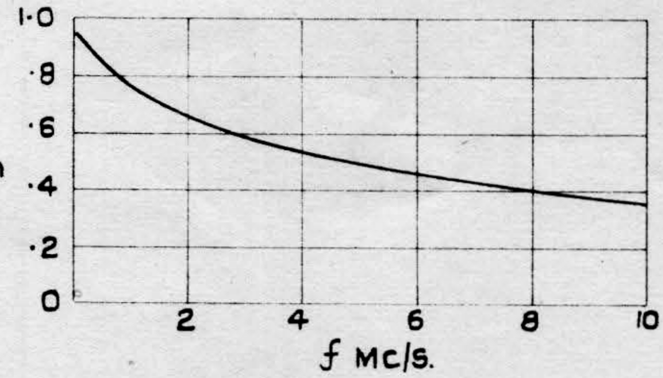


FIG. 13.

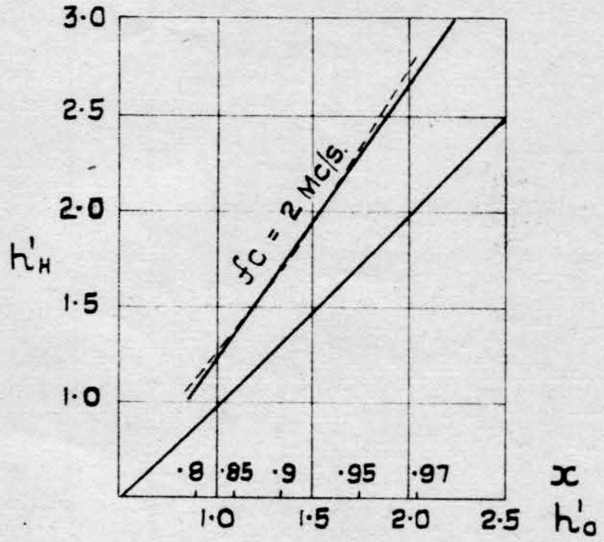


FIG. 14.

RELATIONS OF h'_H TO h'_o

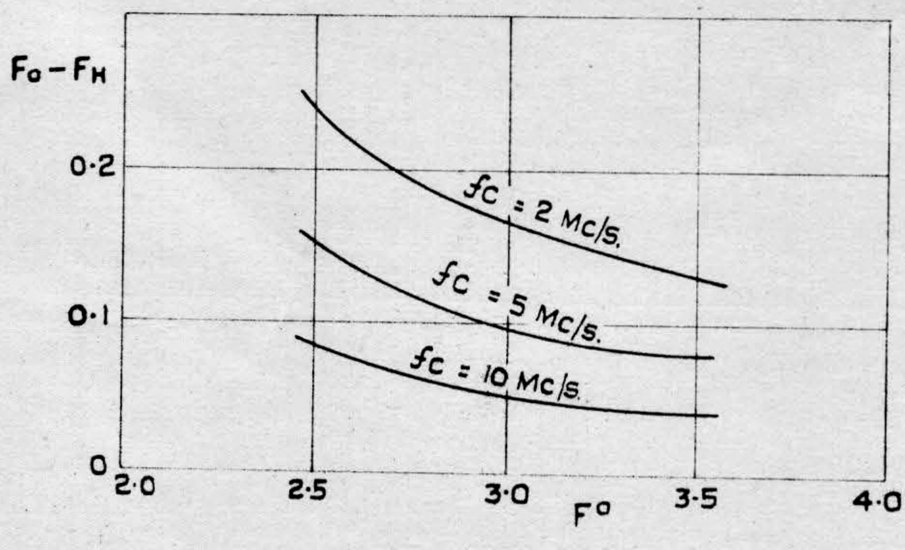


FIG. 15.

COMPARISON
OF
SKIP FACTORS
(2500 KMS.)